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The Plastic Flow of Isotropic Polymers*

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This paper describes a study of the plastic stress-strain behaviour of a number of polymeric materials deformed under different states of stress. Tests were carried out on samples of polymethyl methacrylate, polystyrene, polyethylene terephthalate, polyvinyl chloride, epoxy resin, and high density polyethylene. The extent to which plasticity theory can be used to describe the observed behaviour has been considered and it is concluded that in spite of their viscoelastic behaviour and the large elastic strains prior to yield plasticity theory can be applied provided that stresses are expressed as true stresses at yield. The most appropriate yield criterion for PMMA is a von Mises criterion modified so that the yield stress varies linearly with the hydrostatic component of the stress tensor. For PS a modified Tresca criterion is more appropriate.

The inclination of the bands of local shear deformation that form in these materials is not at 45° to the directions of principal stress. In most cases this apparent deviation appears to be entirely due to elastic recovery on unloading, but for polystyrene there is an extra deviation probably due to the occurrence of small plastic volume changes during yield.

1. Introduction

Any attempt to discuss the flow stress and plastic deformation of polymeric materials in terms of plasticity theory is complicated by the fact that their behaviour is markedly different from that of any "ideal" rigid-plastic model material. A yield point can generally be defined satisfactorily as the point at which the polymer starts to flow at constant stress [1, 2] but polymers can undergo viscoelastic strains of up to 10% before they yield plastically. These elastic strains may produce significant changes in volume and it is possible that further changes in volume take place during plastic flow [3]. Tensile tests on samples under hydrostatic pressure [4-7] and yield stress measurements in tension and compression, [8, 9] and under conditions of combined stress [2, 10] have shown that the yield stress of many polymers is pressure-dependent. The zones of plastic deformation observed at stress concentrations in deformed polymers are not at 45° to the principle axes of stress and it has been suggested that this deviation is connected with the pressure-dependence of the yield stress [8] and volume changes during yield [9].

This paper describes a study of the stress-strain behaviour of a range of polymeric materials tested under different states of stress. Their observed behaviour has been compared with the predicted behaviour of a number of model materials obeying different yield criteria. The question as to which criterion most suitably describes the observed behaviour and the extent to which the observed behaviour approaches that of an ideal plastic material has been considered in detail.

2. Experimental Procedure

2.1. Materials and Sample Preparation

Tests were carried out on seven different polymeric materials.

Polymethyl Methacrylate (PMMA) The material used was I.C.I. "Perspex" acrylic sheet and rod. Samples were laid on a bed of talc and annealed in an air oven at 110°C for 24 h and then cooled slowly over a further 24 h. The cylindrical specimens used for the experiments described in section 2.3 were cut from commercial rod,

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annealed, and afterwards returned to a circular cross section by machining. The weight-average molecular weight was 2.3×10^5 u, and there was no plasticiser present.

Polystyrene (PS) This was Shell "Carinex" QP crystal supplied in granular form. Sheet specimens 3 mm thick were compression moulded. Before testing they were annealed at 100°C using the same technique as was used for PMMA. The weight-average molecular weight was about 10^5 u and there was a small amount of plasticiser present ($< 1\%$).

Polyethylene Terephthalate (PET) Samples were cut from a 1.5mm-thick sheet of I.C.I. "Melinex" precursor. This is an amorphous glassy material with no detectable crystallinity and a density of 1.337 g/cc at 23°C.

Polyvinyl Chloride (PVC) Samples were cut from sheets of I.C.I. "Darvic 024" which is a clear transparent unplasticised grade with a slightly bluish tinge. This particular material does not strain-whiten when deformed in tension as do other grades of rigid PVC, such as "Darvic 110", which contain small particulate additives.

Epoxy Resins Sheets of epoxy resin were very kindly prepared for us by Ciba (ARL) Ltd at Duxford. Two types of material were tested. The first (Epoxy I) was "Ciba MY 750" epoxy resin with 20 parts by weight plasticiser "DY 040". The second (Epoxy II) consisted of the same resin with 40 parts by weight of plasticiser. The material was cast into sheets 2 mm thick and specimens were cut from these sheets.

High Density Polyethylene (HDPE) The material used was "Rigidex 50" which is a linear polyethylene with a density of 0.96 g/cc and a melt flow index of 5. Sheets approximately 1.5 mm thick were prepared by compression moulding.

2.2. The Plane Strain Compression Test with Applied Tension

A detailed account of the experimental technique used for this test, together with a discussion of the necessary corrections, has been given elsewhere [2, 11] and only a brief description will be given here. The testing arrangement is shown schematically in fig. 1. Sheet specimens were compressed in plane strain between lubricated dies. The procedure used was to apply a tensile load, L_2 in the σ_2 direction using a hydraulic system and then to load the specimen to yield by compression in the σ_1 direction using a Tensometer E 10000 kg tensile testing machine. The separation of the dies (h in fig. 1) was recorded continuously during the test so that the strain in the σ_1 direction, and hence the true-stress value of σ_2, could be calculated. In this way values of σ_1 at yield could be determined for different values of σ_2. In analysing the stresses acting on the sample during yielding in polymeric materials it is important to distinguish between *true stress* (force per unit area at yield) and *nominal stress* (force at yield per unit initial area). This distinction is not as important for metals at their yield point since elastic strains are small, but for polymers the elastic strains in the material at yield can be of the order of 10%. In the following analysis stresses are *true stresses* in all cases.

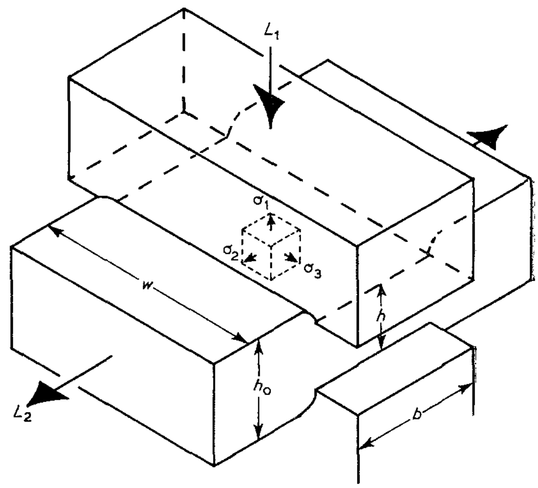

Figure 1 Schematic view of testing arrangement. L_1 and L_2 are the loads applied in the directions indicated. $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses acting on the material under the dies. In terms of *true stress* $\sigma_1 = -L_1/wb$ and $\sigma_2 = L_2/wh$ where h is the instantaneous separation of the dies. $b \sim 6.3$ mm, $w \sim 36$ mm, $h_0 \sim 2$ mm.

It was necessary to consider three corrections in order to determine exact values of σ_1. Firstly, for the polished and lubricated dies used, friction has already been shown to introduce a negligible error in the measured values of σ_1 for PMMA [2] and the same was found to be true for the other materials tested [11]. Secondly it has been shown [12] that the measured compressive load in the plane strain compression test can be increased by the elastic restraining force of the undeformed material outside the dies (edge effect). We found this effect to be important only for PMMA and the results quoted have been corrected for it. Thirdly, the edges of the specimen under the dies are not fully constrained. They do not deform in plane strain and tend to bulge outwards. For the wide sheet specimens used we found that this produced a negligible error in the calculated values of σ_1.

2.3. The Yield Stress of PMMA in Uniaxial Compression

To determine the uniaxial compressive yield stress for PMMA, flat-ended cylindrical specimens approximately 9 mm in diameter with diameter-to-height ratios between 0.2 and 2 were compressed between lubricated parallel dies. During the test there was no detectable barreling of the specimen and the yield stress was taken as the maximum in the stress-strain curve as was done for the plane strain compression tests (see section 3.1). The unconstrained compressive yield stress was obtained by plotting calculated values of the yield stress against diameter-to-height ratio and extrapolating to zero diameter.

2.4. Observation of the Mode of Deformation

In the plane strain compression test illustrated in fig. 1 strains are confined to the σ_1-σ_2 plane and after deformation it was possible to cut sections parallel to this plane and to examine them in polarised light. Sections were cut out roughly with a saw, ground down to a thickness of approximately 0.5 mm on emery paper flooded with water, and finally polished using "Silvo" metal polish and a soft cloth.

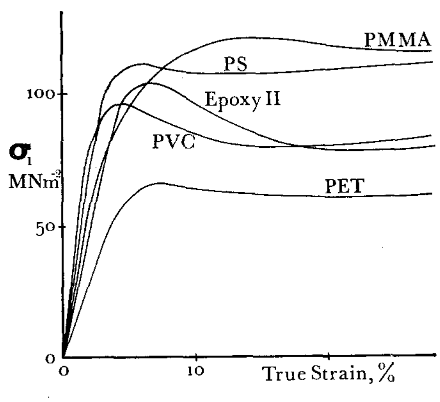

Figure 2 Stress-strain curves determined in plane strain compression.

3. Results

3.1. Yield Stress Measurements in Plane Strain

A stress-strain curve determined in plane strain compression ($\sigma_2 = 0$) is shown in fig. 2 for five of the materials tested. All curves show the same general features. The stress rises to a maximum value (the upper yield stress) at which point residual plastic deformation first becomes detectable on unloading [11]. As the strain increases further strain softening occurs and the flow stress falls. At larger strains the flow stress increases again as the material becomes oriented. We have used the value of the upper yield stress for the purpose of defining a yield criterion. On applying the additional transverse stress, σ_2, similar curves were obtained and the yield stress could always be defined in the same way. For the more brittle materials such as PMMA and PS it was not possible to apply a tension, σ_2, of more than a certain amount or fracture occurred before yielding.

The results in fig. 2 have been obtained using a constant cross-head speed. A test carried out at constant cross-head speed does not give a constant specimen strain rate. While the stress is rising the specimen strain rate is less than that computed from the cross-head speed because of the elasticity of the loading system. However, at the yield point where the stress is constant the specimen strain rate corresponds to that computed from the cross-head speed, and comparative tests carried out at a constant strain rate (obtained by continuously adjusting the cross-head speed manually) showed that the measured yield stress depended only on the strain rate at the yield point [11].

Values of the compressive stress at the upper yield point determined for different values of applied tensile stress, σ_2, are plotted in fig. 3, for all the materials tested with the exception of HDPE and PMMA. The lines drawn on the figure are the best straight lines through the experimental points. The results obtained for PMMA have already been published [2] and the best straight line is shown dashed in fig. 3.

3.2. The Ratio of the Uniaxial Compressive Yield Stress to the Plane Strain Compressive Yield Stress for PMMA and PS

The uniaxial compressive yield stress of PMMA was determined as described in section 2.3 to be 106.0 ± 7.5 MNm^{-2} at a strain rate of 0.07 min^{-1}. In order to make a direct comparison between the uniaxial and plane strain compressive yield stresses it was necessary to measure the plane strain yield stress on the same material as

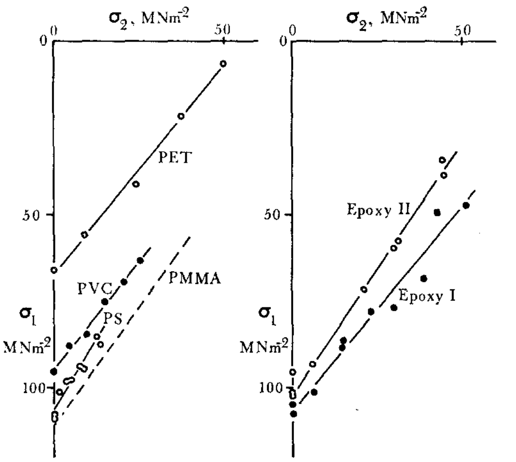

Figure 3 Values of the compressive stress, σ_1, at yield plotted against the applied tensile stress, σ_2.

was used for the uniaxial yield stress measurements. This was done on sheet specimens approximately 3 mm thick cut out of the PMMA rod used for the uniaxial stress measurements. Using the same strain rate as before the plane strain compressive yield stress was found to be 135.8 $\pm$ 1.6 MNm^{-2}. The plane strain yield stress is higher than the uniaxial yield stress, and the ratio of the two figures is 1.28 $\pm$ 0.11. This result will be discussed further in section 5.3.

A value for the uniaxial compressive yield stress of PS of 106 $\pm$ 4 MNm^{-2} at a strain rate of 0.07 min^{-1} was determined using a technique similar to that described for PMMA in section 2.3. If this is taken with a value of 110 MNm^{-2} for the yield stress of PS in plane strain compression determined on samples of the same material, the plane strain value is again slightly higher than the uniaxial one and the ratio of the two values is 1.04 $\pm$ 0.06.

3.3. Structural Observations

Sections cut from each material were examined in transmission between crossed polars. Of the materials tested PMMA and PVC showed broad diffuse regions of deformation that will be referred to as deformation zones, while PS and PET contained thin well-defined strongly birefringent regions containing a high shear strain that will be referred to as deformation bands. The characteristics of these two types of deformation have been discussed in detail elsewhere [13]. The epoxy resins contained zones similar to those observed in PMMA but less diffuse (see fig. 4c of reference [2]).

The zones or bands always formed on planes whose normals were inclined at more than 45° to the compression direction. The inclination of the bands at the instant they form will differ from the inclination measured after unloading because of the large elastic recovery on unloading. If α is the angle between the shear plane normal and the compression direction in the relaxed specimen, then the value of α at the yield point when the bands form, α', is given by

$$\tan \alpha' = \tan \alpha \frac{(1 + \epsilon_1)}{(1 + \epsilon_2)}$$

where ϵ_1 and ϵ_2 are the nominal elastic strains in the specimen at yield. Results are given in table I. Values of ϵ_1 were taken directly from the stress-strain curves and values of ϵ_2 were deduced by assuming a Poisson's ratio of 0.4. Because of the errors in the measured value of α this assumption is sufficiently accurate. The maximum possible value of ϵ_2 is $-\ \epsilon_1$ (a Poisson's ratio of 0.5) and values of α' calculated on this basis are given in brackets.

It can be seen from table I that for all materials with the exception of PS, α' can be 45° within the experimental error. For PS α' is very definitely greater than 45°. Shear zones in PS form before the upper yield point [9] and this effect will increase α' further.

Fig. 4 is a section cut from a PVC sheet bent in four-point loading. The top surface of the sample in the figure was under compression while the bottom was under tension. Shear zones can be seen propagating from scratches on both

TABLE I

Material	ϵ_1 at yield	α (measured)	α' (corrected)
PMMA	− 0.13	52.9 $\pm$ 2.1	46.6° (45.5°) $\pm$ 2.1
PS	− 0.06	52.7 $\pm$ 0.7	*49.8° (49.2°) $\pm$ 0.7*
PET	− 0.07	48.0 $\pm$ 1.5	44.6° (44.0°) $\pm$ 1.5
PVC	− 0.05	48.8 $\pm$ 1.9	46.4° (45.8°) $\pm$ 1.9
Epoxy II	− 0.07	50.4 $\pm$ 1.5	47.0° (46.4°) $\pm$ 1.5

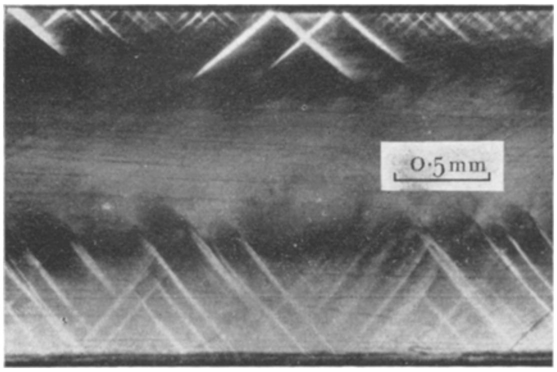

Figure 4 Section cut from a PVC sheet deformed in four-point bending viewed between crossed polars. The top surface was in compression.

surfaces, the zones on the compression surface where the extension direction is vertical being at a different inclination to those on the tension surface where the extension direction is horizontal. It is also apparent that zones have propagated further from the tension surface than from the compression surface, demonstrating directly that the flow stress in tension is less than that in compression.

3.4. Results on High Density Polyethylene

Strip specimens of HDPE were tested in plane strain compression with and without applied tension, and in uniaxial tension. The results are presented in fig. 5.

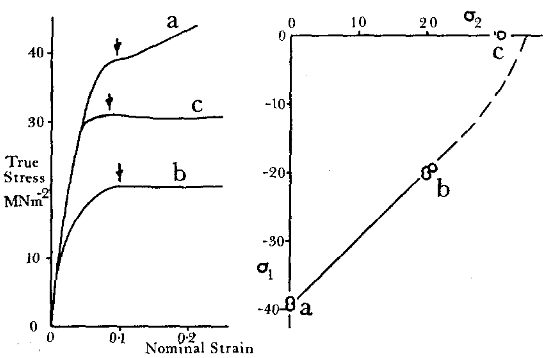

Figure 5 Results obtained on HDPE for tests in (a) plane strain compression, (b) pure shear and (c) uniaxial tension.

Curve a in fig. 5 is a stress-strain curve determined in plane strain compression with $w = 30$ mm and $b = 6.12$ mm (see fig. 1). It can be seen that at yield there is a kink in the curve followed by a slight inflection. The yield stress was taken as the stress at this point as indicated by the arrow.

Curve c is a stress-strain curve determined in uniaxial tension on a strip specimen approximately 15 mm wide. The specimen drew down to form a neck and the width and thickness of the necked region were monitored continuously during the test. The nominal strain in the neck was calculated from the reduction in cross-sectional area, and the true stress was calculated from the load and the instantaneous cross-sectional area. The true stress rose to a maximum value, dropped slightly, and then increased, and the yield stress was taken to correspond to this initial maximum value as shown.

Curve b is the stress-strain curve determined in plane strain compression with a constant load applied in the σ_2 direction. Flow occurred at a constant compressive stress, and the yield point was taken to correspond to 10% strain as indicated for the purposes of calculating the true-stress value of σ_2 at yield.

Yield-stress values obtained from a number of such tests are also plotted in fig. 5. These results are discussed in section 5.4.

4. The Yield Behaviour of Ideal Plastic Materials

In this section the yield behaviour of a number of ideal plastic materials is discussed in detail for later comparison in section 5 with the experimental results. The extent to which polymers can be considered to behave as ideal plastic materials is considered in section 5.1.

4.1. Yield Criteria

The criterion that has been found most appropriate to describe the onset of plastic flow in metals is that due to von Mises, which can be written in terms of the principal stresses as

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6k^2 \quad (1)$$

The quantity k is a constant. The criterion predicts that yield occurs when the elastic shear strain energy density in the material reaches a critical value. A simpler criterion is that due to Tresca which postulates that yield will occur when the resolved shear stress on any plane in the material reaches a critical value.

$$\text{Maximum shear stress} = k \quad (2)$$

For both criteria the constant k is the shear stress for flow in pure shear, and the flow stress is independent of the hydrostatic component of the stress system and depends only on the deviatoric component.

For many polymers the flow stress increases when the material is under a hydrostatic pressure and two different ways have been suggested of modifying the yield criterion to take this effect into account. It has been suggested [2, 9] that by analogy with the Coulomb yield criterion [14] the value of the constant k increases linearly with p_n, the normal pressure on the shear plane ($p_n = -\sigma_n$ where σ_n is the normal stress), leading to a relation of the form

$$k = k_0 - \mu\sigma_n = k_0 + \mu p_n \qquad (3)$$

where μ and k_0 are constants. A relation of this type can be, but need not be, due to a volume increase during yield (see section 4.3.3). The "shear plane" is the plane in the material on which the shear stress first reaches the critical value for yield defined by equation (3) [14].

A second possibility suggested by Schofield and Wroth for soils [15] is that the value of k depends on the hydrostatic component of the stress system

$$k = k_0 + \mu P, \quad P = -\tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad (4)$$

The physical argument behind this latter assumption is that the hydrostatic stress changes the *state* of the material and produces a structure that has a higher flow stress but is otherwise a well-behaved plastic material. For example it deforms at constant volume.

4.2. Application of the Yield Criteria

If the two forms of yield criterion, equations (1) and (2) are combined with the two forms of pressure dependence, equations (3) and (4), there are four possible combinations. However only three of these are physically realistic. For deformation according to the von Mises criterion it is not reasonable to define a shear plane (see section 4.3) so that it is difficult to justify the use of equation (3) with equation (1). The three realistic combinations are:

(*a*) Yield according to equation (1) with a k value determined by equation (4). This we will refer to as a *modified von Mises criterion.*

(*b*) Yield according to equation (2) with the value of k determined by equation (3). This is a criterion that has been suggested for soils [14] and we will refer to it as a *Mohr-Coulomb criterion.*

(*c*) Yield according to equation (2) with the value of k determined by equation (4). This we will refer to as a *modified Tresca criterion.*

In sections 4.2.1 to 4.2.3 the yield stresses predicted by these three criteria are given for a number of simple mechanical tests.

4.2.1. Plane Strain Compression in the σ_1 Direction; $\sigma_2 = 0$

In the plane strain compression test (fig. 1) the material under the dies is prevented from expanding in the direction perpendicular to σ_1 and σ_2 so that during yielding the strain rate in this direction is zero. It follows from the Lévy-Mises equation for plastic flow [16] that the restraining stress in this direction, σ_3, is given by $\sigma_3 = \frac{1}{2}(\sigma_1 + \sigma_2)$.

If σ_3 has this value then on both a modified von Mises criterion and a modified Tresca criterion yield will occur when

$$\sigma_1 = -2k_0/(1 - \mu) \qquad (5)$$

On the Mohr-Coulomb yield criterion yield will occur when

$$\sigma_1 = -2k_0/(\sqrt{(1 + \mu^2)} - \mu) \qquad (6)$$

For small values of μ equation (6) approximates to equation (5).

4.2.2. Pure Shear (Plane Strain); $\sigma_1 = -\sigma_2$, $\sigma_3 = 0$

In a test in pure shear the hydrostatic component of the stress system is zero so that on both a modified von Mises and a modified Tresca criterion yield will occur when

$$\sigma_1 = -\sigma_2 = -k_0 \qquad (7)$$

On the Mohr-Coulomb yield criterion yield will occur when

$$\sigma_1 = -\sigma_2 = -k_0/\sqrt{(1 + \mu^2)} \qquad (8)$$

For small values of μ equation (8) approximates to equation (7).

It follows from equation (7) that the value of k_0 can be obtained directly from the flow stress in a pure shear test whatever yield criterion the material obeys. The value of μ can then be obtained from the measured yield stress in plane strain compression using equation (5).

4.2.3. Uniaxial Compression in the σ_1 Direction; $\sigma_2 = \sigma_3 = 0$

From the tests described above it is possible to determine numerical values of k_0 and μ but it is not possible to determine which yield criterion is most appropriate for a given material since the flow-stress values predicted from all three

TABLE II

Yield Criterion	σ_u	σ_{ps}/σ_u	
		$\mu = 0.16$	$\mu = 0.25$
modified von Mises	$-\dfrac{\sqrt{3}\,k_0}{1 - \mu/\sqrt{3}}$	1.25	1.32
Mohr-Coulomb	$-\dfrac{2\,k_0}{\sqrt{(1+\mu^2)} - \mu}$	1	1
modified Tresca	$-\dfrac{2\,k_0}{1 - 2\mu/3}$	1.06	1.11

criteria are essentially identical. This is not the case for tests in uniaxial compression. In table II expressions for the uniaxial compressive yield stress, σ_u, are given for the three criteria. The yield stresses in uniaxial tension are obtained by changing the signs in these expressions. For instance, the tensile yield stress on the modified von Mises criterion is $+\sqrt{3}\,k_0/(1 + \mu/\sqrt{3})$.

A practical method of determining which criterion is most appropriate for a given material is to compare the yield stress in plane strain compression, σ_{ps}, with the yield stress in uniaxial compression. In table II numerical values of the ratio σ_{ps}/σ_u are given for μ-values of 0.16 and 0.25.

4.3. Volume Changes and the Inclination of the Shear Plane

In sections 4.1 and 4.2 it was implicitly assumed where necessary that deformation occurs by shear on planes of maximum shear stress. This assumption needs to be examined more critically, particularly for the case of materials that change their volume during plastic flow.

The first important point to be made is that for any homogeneous deformation it is in general not necessary or possible to define a shear plane. The material has extended in the tensile direction and contracted laterally but if the deformation is homogeneous there is no particular plane in the structure on which shear can be said to have occurred. The concept of a shear plane is not necessary, although it may be introduced as a theoretical convenience as is done in the Tresca yield criterion.

It is only if locally in the material the deformation is larger than elsewhere, due for example to some stress concentration which produces locally slightly higher strains, that a more or less well defined shear plane becomes apparent. Such "visible shear planes" (or, in the case of soils, *failure planes*), are always at 45° to the principal axes of stress in materials which deform at constant volume such as metals, but for materials which change their volume during plastic deformation this is not necessarily so. For both plastics [2, 3, 8, 9], and soils, [14] volume changes at yield and the deviation of the "visible shear plane" from 45° have been noted and interpreted.

In the following sections the conditions necessary for the development of "visible shear planes" in two types of ideal plastic material are considered. The two types are (*a*) a material that deforms plastically at constant volume, and (*b*) a material that undergoes a continuous volume dilatation during plastic deformation. For both materials it is assumed that plastic flow occurs at a constant stress, the yield stress, and that the principal axes of stress and plastic strain rate coincide (St Venant's Principle).

A general condition to be satisfied is that any region where locally the plastic strain rate is higher than elsewhere (a developing visible shear plane) must remain coherent with the rest of the material. A second condition is that the local deformation must have its components of principal strain rate in the same ratio to each other as those of the general deformation (i.e. the local deformation must cause the same change of shape as does the general deformation). This condition follows from the Lévy-Mises equations of plastic flow for an ideal plastic material. The ratios are fixed by the state of stress at yield. For such a material all rates of flow can occur at the flow stress so there is no basic difficulty about a local deformation developing provided both the above conditions are satisfied. If both conditions are to be satisfied then the local deformation can only occur *on planes in which during general yield the extension rate is zero.* For local deformations on other planes either a discontinuity would appear between the local deforming region and the rest of the material or the ratio of the principal strain rates would have to change locally as the deformation developed.

4.3.1. *Extension in Plane Strain at Constant Volume*

If a material which deforms plastically at constant volume is being extended in plane strain, then the situation is as shown schematically in fig. 6a. Since ϵ_3 is zero, $\epsilon_2 = -\epsilon_1$, and the relevant Mohr circle of strain rate is centred on the origin.

The only planes in which the extension rate is

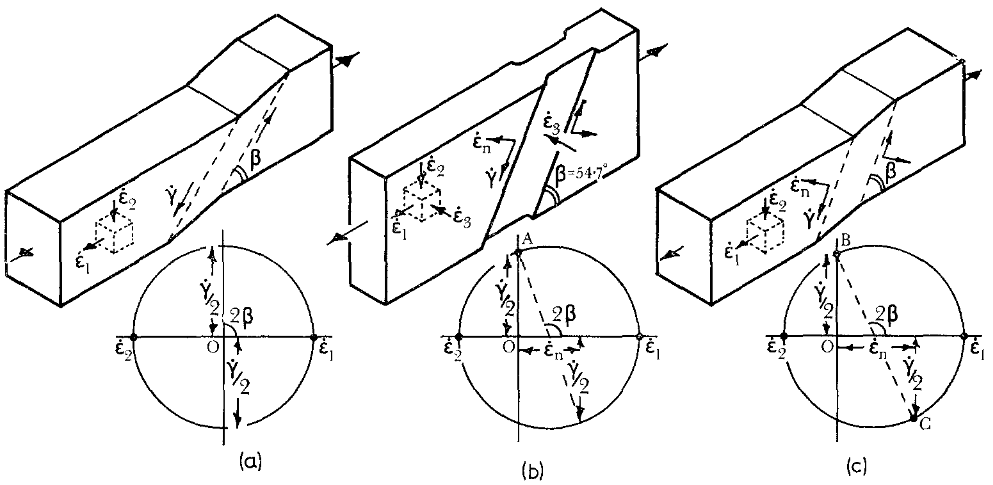

Figure 6 Schematic representation of strain-rate components together with the appropriate Mohr circle of strain rate for (a) extension in plane strain, (b) uniaxial extension and (c) plane strain extension with a volume dilatation.

zero are those inclined at 45° to the extension direction Moreover deformation in *simple shear* of a band of material at this inclination at an angular shear strain rate of $\dot{\gamma}(= \dot{\epsilon}_1 - \dot{\epsilon}_2)$ is equivalent to the *pure shear* deformation imposed on the sample. (The simple shear involves an additional rotation rate of $\dot{\gamma}/2$, but for an ideal plastic material this does not affect the argument).* Consequently, since in an ideal plastic material all strain rates can occur at the same stress, local deformation bands can develop freely on these planes. The same argument applies to a compression in plane strain.

4.3.2. Uniaxial Extension at Constant Volume

Fig. 6b illustrates the deformation of a thin strip of material in uniaxial extension. For uniaxial extension $\dot{\epsilon}_2 = \dot{\epsilon}_3 = -\dot{\epsilon}_1/2$ and these strain rate components define the relevant Mohr circle shown in the figure. The only possible local deformation that can develop with these same strain-rate components (and therefore at the same stress) is the thinning and shearing of a band of material at 54°44′ to the tensile axis as illustrated. The inclination of the band is represented by the point A on Mohr's circle and the precise value of 54° 44′ ($\cos 2\beta = -1/3$) follows from the geometry [17].

For such a zone to form the zone width must be comparable with or greater than the thickness of the strip so that local thinning is possible. If the zone were thin compared with the thickness of the strip, then local thinning would not be possible ($\dot{\epsilon}_3 \rightarrow 0$) and the only local deformation capable of developing would be simple shear in plane strain as described in section 4.3.1. Extension in plane strain requires a slightly higher applied stress than uniaxial extension because of the plane strain condition which imposes different relative components of strain rate. There is a finite σ_3 acting to maintain the plane strain condition and this provides the constraint. Consequently during uniaxial extension zones of local deformation can only develop if locally the stress is raised to the plane strain yield stress. If such zones develop they are on planes at 45° to the tensile axis for the reasons given in section 4.3.1.

4.3.3. Extension in Plane Strain with a Volume Dilatation

Fig. 6c illustrates the situation when a material which dilates during plastic deformation is extended in plane strain. The volumetric strain rate, $\dot{v}/v = ((\mathrm{d}v/\mathrm{d}t)/v)$, is given by $\dot{v}/v = \dot{\epsilon}_1 + \dot{\epsilon}_2 + \dot{\epsilon}_3$. Since $\dot{\epsilon}_3$ is zero, $\dot{\epsilon}_2 = \dot{v}/v - \dot{\epsilon}_1$, and the centre of the Mohr's circle is displaced from the origin by $\dot{v}/2v$.

A plane in which the linear strain rate is zero is represented by a point such as B. If a band of material parallel to this plane deforms in simple shear at a strain rate of $\dot{\gamma}$ and at the same time

*Some real materials however exhibit different stress-strain characteristics in simple shear and pure shear at large strains.

increases in thickness so that there is a linear strain rate normal to the plane of the band of $\dot{\epsilon}_n (= \dot{v}/v)$, then (ignoring the rotation produced by a simple shear) this deformation could be represented by points B and C on the Mohr circle in which case it would have the same strain rate components as the original deformation and could occur at the same stress.

The inclination of the planes on which such deformation bands form depends on the ratio $\dot{\epsilon}_n/\dot{\gamma}$. If the band is at angle β to the extension direction then it can be seen from fig. 6c that $\dot{\epsilon}_n/\dot{\gamma} = \dot{v}/v\dot{\gamma} = -\cot 2\beta$. It follows that β will be greater than 45° if $\dot{\epsilon}_n/\dot{\gamma}$ is positive. It has been found experimentally that failure planes in soils form parallel to the directions of zero extension rate and has been argued that this occurs for the reason given here, namely that it is the only local deformation possible having the same components of strain rate [18].

4.4. Volume Changes and Pressure-Dependent Yielding

In earlier treatments [14] it was held that the inclination of the failure zones was related directly to the pressure-dependence of the yield stress. This misunderstanding arose because for an ideal dilating plastic material such as the one described in section 4.3.3. the yield stress does indeed depend on the pressure. If such a material is deformed in simple shear the applied shear stress, τ, has to do work against the shear yield stress of the material, τ_0; but work is also done by any tensile stress, σ_n, acting normal to the shear plane because the material is dilating ($\dot{\epsilon}_n = \dot{v}/v$). Since the work done per unit volume is the stress multiplied by the strain rate, $\tau\dot{\gamma} = (\tau_0\dot{\gamma} - \sigma_n\dot{v}/v)$ and hence

$$\tau = \tau_0 - (\dot{v}/v\dot{\gamma})\,\sigma_n \qquad (9)$$

This equation is analogous to equation (3) with μ equal to $\dot{v}/v\dot{\gamma}$. What has only relatively recently been pointed out [15] is that it is possible to have a material whose yield stress depends on pressure not because it dilates plastically but because the pressure changes the properties (and hence the yield stress) of the material. If such a material deforms plastically at constant volume then it will behave as described in sections 4.3.1 and 4.3.2. Any plastic volume dilatation will make a contribution to the pressure dependence of the yield stress over and above that due to the effect of pressure on the properties and will alter the inclination of deformation bands as described in section 4.3.3.

5. Discussion

5.1. The Application of Plasticity Theory to Polymers

It was necessary to make a number of implicit or explicit assumptions in order to obtain the results described in section 4 and the extent to which these assumptions are valid for polymeric materials must be considered.

In this work an attempt has been made to define a yield criterion and to analyse the plastic behaviour only for the stress state at the yield point where plastic deformation starts and continues for a short time at constant stress. Once a polymeric material has undergone an appreciable plastic strain its mechanical properties become highly anisotropic and it exhibits a strong Bauschinger effect [19]. In addition many polymers strain-harden rapidly.

Polymers undergo large viscoelastic strains as they are loaded to their yield point. Elastic strains produced in a material before the yield point is reached do not affect the arguments of plasticity theory provided that during yielding the elastic strain is constant [16]. This will be the case as long as flow takes place at constant stress. However an effect of a large elastic strain is to distort the material and if the strains are large to make it markedly anisotropic by the time plastic deformation starts to occur. The arguments about plastic behaviour and in particular the arguments about the inclination of deformation bands (section 4.3) assumed the material to be isotropic. The inclination of shear bands is affected by anisotropy [19].

It was suggested by us in an earlier paper [2] that St Venant's Principle (section 4.3) would not be valid for polymers since their yield stress depends on pressure and yet they appear to deform at constant volume and so they must be the "frictional materials" described by de Jong [20]. However, the critical-state argument (section 4.1) used by Schofield and Wroth makes it unnecessary to consider them as frictional materials, and consequently St Venant's Principle is expected to be obeyed as it is found experimentally to be obeyed for soils [21], which in de Jong's sense are not frictional materials either.

Provided the effects of anisotropy due both to elastic deformation before yield and plastic deformation after yield can be neglected the results described in section 4 should apply to polymers. In practice this means only to initially isotropic polymers at and a little beyond the point where they first start to flow plastically. It should be emphasised again that stresses must be expressed as true stresses based on the instantaneous shape of the sample during flow.

5.2. Values of k_0 and μ

The data given in figs. 3 and 5 can be used to derive values of k_0 and μ whichever of the criteria described in section 4.1 are appropriate. The value of k_0 can be derived from the value of σ_1 and σ_2 at the pure shear point (equation (7)). If k_0 is known the value of μ can be derived from the yield stress in plane strain compression using equation (5) which holds approximately for all three yield criteria. The results for all the materials tested are given in table III.

TABLE III

	k_0, MNm^{-2}	μ	Strain rate, min^{-1}
PMMA	47.4 ± 1.1	0.158 ± .02	0.128
PS	40.0 ± 3.4	0.25 ± .05	0.083
PET	31.0 ± 1.9	0.09 ± .02	0.288
PVC	42.0 ± 1.5	0.11 ± .02	0.121
Epoxy I	49.0 ± 4.4	0.09 ± .03	0.133
Epoxy II	42.0 ± 1.8	0.19 ± .03	0.133
HDPE	17.4 ± 0.6	< 0.05	0.138

The values of μ in table III are consistent with values obtained by other workers using different techniques. A value of 0.15 has been obtained for PMMA tubes tested under combined tension and torsion [10]. Careful experiments on the torsion of tubes under high hydrostatic pressure have given μ-values of 0.204 for PMMA, 0.075 for PET and 0.034 for HDPE [7].

5.3. Choice of a Yield Criterion for PMMA and PS

It is possible to make a choice as to which of the yield criteria discussed in section 4.1 is the most appropriate for PMMA. In section 3.2 results were presented which showed that the ratio for PMMA of the plane strain to the uniaxial compressive yield stress was 1.28. In table II this ratio has been evaluated for a μ-value of 0.16 and it is seen that the criterion most consistent with the observed ratio is a modified von Mises criterion. This is the form suggested by Sternstein *et al* [10] and is also the form used by Rabinowitz, Ward, and Parry [7] to interpret their results on PMMA. The observation that deformation bands in PMMA form at 45° to the directions of principal stress (section 3.3) is to be expected for a material which obeys such a criterion (section 4.3.1).

For PS the value of μ is 0.25 and the ratio of the plane strain to the uniaxial yield stress is much closer to unity (1.04). Referring to table II it can be seen that either a Mohr-Coulomb or a modified Tresca criterion is appropriate. From the value of the yield-stress ratio alone it is not possible to decide which. It has been shown by Taylor [22] that if a material deforms by the formation of thin shear bands (Lüders bands) as does PS then even if the deforming material inside the bands obeys a von Mises criterion the material as a whole will obey a Tresca criterion. If plastics in general obey a modified von Mises criterion as does PMMA then PS might be expected to obey a modified Tresca criterion because it deforms by the formation of shear bands. However, this would not explain the observed deviation of the shear bands from 45° (section 3.3). It was shown in section 4.4 that if a material dilated continuously during plastic flow then the flow stress will depend on the normal stress on the shear plane so that it will obey a Mohr-Coulomb criterion. The inclination of the shear bands will deviate from 45° for the reasons given in section 4.3.3. However, there are difficulties in accepting that a continuous dilatation takes place in PS during flow. To explain the observed deviation of the shear bands from 45° it is necessary to postulate a value of $\dot{v}/v\dot{\gamma}$ of about 0.15. The plastic shear strain in the deformation bands is known to be of the order of 2 [13] which would imply a volume dilatation of $dv/v = 0.3$ or 30% if $\dot{v}/v\dot{\gamma}$ remains constant throughout the deformation process. From density measurements on the deformed material it can be shown that if there is any permanent plastic dilatation in the deformation bands, it is less than 0.5%. In any case the idea of a volume dilatation of 30% during a compression test is difficult to accept.

A possible explanation of the observed behaviour of PS is that there is a small but finite dilatation when plastic deformation starts, sufficient to give an initial value of $\dot{v}/v\dot{\gamma}$ of about 0.15. This would ensure that the inclination of the deformation bands deviates from 45° as is observed. Deformation then can proceed at constant volume ($\dot{v}/v\dot{\gamma} = 0$) and the observed pressure dependence of the flow stress would then be due to the effect of the hydrostatic pressure on the properties of the material (equation (4)) and not to any volume change. Such a small initial dilatation would make a contribution to the pressure dependence of the flow stress but the magnitude of the contribution will depend on the *average* value of $\dot{v}/v\dot{\gamma}$ during flow, not the initial value, and so would be small. If this is the case then the yield stress of PS might be expected to obey a modified Tresca criterion.

5.4. PVC, PET, Epoxy Resins and HDPE

If the ideas and results discussed in the last section can be applied to polymers in general then it might be expected that all polymers which deform relatively homogeneously (PVC, Epoxy resins and HDPE) should obey a modified von Mises criterion while those that deform inhomogeneously by the formation of shear bands (PET) should obey a modified Tresca criterion. We have not carried out an experiment to determine which is most appropriate for materials other than PMMA and PS.

PVC behaves plastically in a manner very similar to PMMA. The two epoxy resins again behave similarly; the effect of added plasticiser appears to be to increase μ.

For HDPE μ appears to be small. For the tests in plain strain compression and pure shear summarised in fig. 5 the material is deforming in plane strain. However for the test in uniaxial tension the plane strain condition is relaxed and if the material obeys a modified von Mises criterion the yield stress in tension should be given by the point indicated by the dashed extrapolation in fig. 5. The measured value was slightly lower than this predicted value but the difference may be either not significant or due to the voiding (strain whitening) that is observed to occur in PE in a tension test.

5.5. A Model to Describe the Pressure Dependence of the Flow Stress

A number of suggestions have been made as to source of the pressure-dependence of the flow stress in polymeric materials. Argon *et al.* [9] suggested that it is due to the difficulty in pulling a lumpy molecular chain through gaps between adjacent molecules, which depends on the pressure. Duckett *et al.* [23] have modified a theory for the flow stress in amorphous polymers due to Robertson [24] to include a term in which the hydrostatic pressure alters the height of the energy barrier for the molecular flow process.

An early model that describes the flow process in polymeric materials is the stress-activation model suggested by Eyring [25] in which the segment jump rate is controlled by activation over a barrier whose height is reduced in proportion to the applied shear stress.

$$\nu = \nu_0 \exp\left(\frac{V_A \tau}{2\, k_B T}\right) \qquad (10)$$

The quantity ν_0 is the jump rate in the absence of a shear stress, V_A is a material constant, the activation volume, k_B is the Boltzmann constant and T is the absolute temperature.

It is known that the rate constants for molecular processes in polymers, as determined both by mechanical and dielectric measurements, depend on hydrostatic pressure [26] and that the dependence is of the form

$$\nu_0 = A \exp(-xP) \qquad (11)$$

where P is the hydrostatic pressure and A and x are positive constants. Substituting this relation in equation (10) gives

$$\nu = A \exp(-xP) \exp\left(\frac{V_A \tau}{2\, k_B T}\right) \qquad (12)$$

If the flow stress is determined at a constant strain rate it is reasonable to postulate that the segment jump frequency during yielding will be the same in all cases. If ν is a constant equation (12) can be re-written

$$\tau = \left(\frac{2\, k_B T}{V_A}\right) \ln\left(\frac{\nu}{A}\right) + \left[\frac{2\, k_B T x}{V_A}\right] P \qquad (13)$$

This equation predicts a linear variation of shear stress with hydrostatic pressure, and by analogy with equation (4) the term in square brackets equals μ. The predicted value of μ can be estimated. The activation volume, V_A, can be determined from the measured variation of yield stress with strain rate and for PMMA it has a value of 1440 Å^3. Sasabe and Saito [26] have measured the pressure dependence of a dielectric relaxation peak in PMMA which they attribute to a main chain relaxation, and their measured value of x is 0.017 $(\text{MNm}^{-2})^{-1}$. These figures predict a value of μ for PMMA of 0.1 which is of the right order of magnitude.

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